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ANALYTICAL STUDY OF $(3+1)$-DIMENSIONAL ZAKHAROV-KUZNETSOV EQUATION

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Abstract. This article investigates the analytic solution to the (3+1)-dimensional Zakharov-Kuznetsov (ZK) equation, a nonlinear partial differential equation used to model physical phenomena in various fields, particularly plasma physics. We employ the ϕ^6- model expansion method to derive Jacobi's elliptic, hyperbolic, trigonometric, and rational function solutions to the ZK equation. In this study, we utilize various parameter regimes to illustrate the graphical representation of the obtained solutions in both 2-dimensional and 3-dimensional formats, which helps us to understand the physical phenomena further.

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1. Introduction

Over the last few decades, numerous disciplines, including but not limited to economics, biology, chemistry, engineering, and physics, have utilized nonlinear evolution equations (NLEEs) to model nonlinear processes. These equations are a type of partial differential equation that describes how a phenomenon evolves over time under the influence of nonlinear interactions. These equations have a broader class of applications in different areas such as plasma physics, nonlinear optics, fluid dynamics, quantum field theory and solid mechanics. NLEEs investigate fluid flow in various contexts, including turbulent flow modelling and fluid-structure interaction. NLEEs are also used to model metal forming, investigate the mechanical behavior of solids, and model the deformation of materials under various loading circumstances. To gain insight into nonlinear phenomena, one has to find exact solutions to the NLEEs. Exact solutions not only enhance our understanding of physical systems, but they also serve as a benchmark to assess the accuracy, stability, and convergence of numerical algorithms. Consequently, the search for exact solutions of NLEEs has been a major focus of research. However, due to the inherent complexity of nonlinear terms, finding exact solutions to these equations is often challenging. Because of the development of powerful computer algebra systems like Maple and Mathematica, the search for exact solutions to NLEEs has attracted much attention recently. These tools allow researchers to execute complex and time-consuming algebraic calculations more effectively. Our literature review indicates that no single method can universally solve all nonlinear evolution equations, as each of these methods has limited applicability. Although many effective analytical techniques for solving NLEEs have been developed, including the decomposition method [7, 8], the tanh-function method [16, 17], the Hirota's bilinear transformation method [9, 10, 12, 23],

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the sine-cosine method [24, 25], the homogeneous balance method [6, 21], the $\left(\frac{G'}{G}\right)$ expansion method [2, 22], the $\exp(-\phi(\xi))$ expansion method [1, 11, 13, 20] and the new ϕ^6-model expansion method [4, 27, 28, 29].

In 1974, Zakharov and Kuznetsov proposed the ZK equation [5, 14, 19, 26]

$$u_t + A_1 u u_x + A_2 u_{xxx} + A_3(u_{yyx} + u_{zzx}) = 0, \tag{1}$$

where the coefficients A_1, A_2, and A_3 are constants. This equation resulted from their experimental analysis of nonlinear dust acoustic waves through the use of reductive perturbation method for studying magnetized three component dusty plasma with negatively charged dust particles, isothermal ions and electrons. The investigation confirms the existence of three-dimensional solitons in low-pressure magnetized plasma that travel along magnetic fields, as these solitons display decay characteristics in all directions. This revolutionary discovery not only increased our theoretical understanding of nonlinear wave processes in plasma, but it also demonstrated the ZK equation's practical applicability. Researchers have utilized various approaches to derive exact solutions for the ZK equation. In particular, [18] utilized Lie symmetry analysis together with the (G'/G)-expansion method and the extended tanh-function method to discover solitary wave solutions as well as periodic solutions and singular periodic solutions. Using the modified extended direct algebraic method, [15] obtained various soliton solutions, including bell-shaped solitons, anti-bell periodic solitons, dark solitons, bright solitons, and periodic bright-dark solitary waves. Furthermore, in [3], the authors obtain numerous structured and illustrative soliton solutions using the enhanced modified simple equation approach. Soliton solutions are crucial for investigating the dynamics of dust ion acoustic waves. These solutions are also directly relevant to experimental plasma setups, providing valuable insights for laboratory investigations and underscoring the importance of the ZK equation in plasma physics. Recognizing the significance of the ZK equation and their exact solutions, this study aims to use the new ϕ^6-model expansion method to explore Jacobi's elliptic, hyperbolic, trigonometric, and rational function solutions to (1).

The remainder of the paper comprises five sections starting from Section 2, which introduces the detailed method description. Section 3, demonstrates the soliton solutions of (1). In Section 4 the obtained solutions are displayed through 3− D and 2− D graphical representations. In Section 5, we discuss the physical interpretation of the results obtained and the conclusions are discussed in Section 6.

2. Description of the ϕ^6-model expansion method

Consider a general nonlinear partial differential equation of the form

$$G(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt} \ldots) = 0, \tag{2}$$

where G is a polynomial function involving the unknown function u and its partial derivatives of various orders. The function u depends on the variables x, y, z and t.

The fundamental steps of the method are outlined as follows:

Step 1: We utilize the wave transformation

$$u(x, y, z, t) = u(\xi),\ \xi = kx + ly + mz - wt, \tag{3}$$

where k, l ,m and w are non zero constants, to transform (2) into the following nonlinear ordinary differential equation:

$$F(u, u', u'', u''', \ldots) = 0, \tag{4}$$

here F represents a polynomial expression in $u(\xi)$ with their total derivatives.

Step 2: Assuming that solution to (4) is as follows:

$$u(\xi) = \sum_{i=0}^{2M} a_i \phi^i(\xi), \tag{5}$$

where M is a positive integer, and the coefficients a_i $(i = 0...2M)$ are unknown parameters that will be systematically determined later. Meanwhile, the function $\phi(\xi)$ is subject to the following nonlinear ordinary differential equations:

$$\begin{aligned} \phi'^2(\xi) &= b_0 + b_2\phi^2(\xi) + b_4\phi^4(\xi) + b_6\phi^6(\xi), \\ \phi''(\xi) &= b_2\phi(\xi) + 2b_4\phi^3(\xi) + 3b_6\phi^5(\xi), \end{aligned} \tag{6}$$

where b_i $(i = 0, 2, 4, 6)$ are real constants to be found later.

Step 3: The solution to (6) is widely known,

$$\phi(\xi) = \frac{P(\xi)}{\sqrt{fP^2(\xi) + g}}. \tag{7}$$

where $(fP^2(\xi) + g) > 0$ and $P(\xi)$ is a solution of the Jacobian elliptic equation:

$$P'^2(\xi) = c_0 + c_2P^2(\xi) + c_4P^4(\xi), \tag{8}$$

where c_i $(i = 0, 2, 4)$ are constants that need to be found later, while our sources f and g are provided by,

$$f = \frac{b_4(c_2 - b_2)}{(c_2 - b_2)^2 + 3c_0c_4 - 2c_2(c_2 - b_2)}, \tag{9}$$

$$g = \frac{3c_0b_4}{(c_2 - b_2)^2 + 3c_0c_4 - 2c_2(c_2 - b_2)}, \tag{10}$$

under the condition of restriction

$$b_4^2(c_2 - b_2)(9c_0c_4 - (c_2 - b_2)(2c_2 + b_2)) + 3b_6(3c_0c_4 - (c_2^2 - b_2^2))^2 = 0. \tag{11}$$

Step 4: The widely recognized Jacobi's elliptic solutions to (8) are listed in Table 1, where $0 < r < 1$ defines the modulus of the Jacobian elliptic functions. Moreover, trigonometric and hyperbolic functions are the degenerates of these functions when $r \to 0$ and $r \to 1$, respectively, as given in Table 2.

Step 5: Equation (2) can be solved as Jacobi's elliptic function solutions by substituting (7) and (8) into (5).

Table 1

No.	c_0	c_2	c_4	$P(\xi)$
1.	1	$-(1+r^2)$	r^2	$sn(\xi)$ or $cd(\xi)$
2.	$1-r^2$	$2r^2-1$	$-r^2$	$cn(\xi)$
3.	r^2-1	$2-r^2$	-1	$dn(\xi)$
4.	r^2	$-(1+r^2)$	1	$ns(\xi)$ or $dc(\xi)$
5.	$-r^2$	$2r^2-1$	$1-r^2$	$nc(\xi)$
6.	-1	$2-r^2$	$-(1-r^2)$	$nd(\xi)$
7.	1	$2-r^2$	$1-r^2$	$sc(\xi)$
8.	1	$2r^2-1$	$-r^2(1-r^2)$	$sd(\xi)$
9.	$1-r^2$	$2-r^2$	1	$cs(\xi)$
10.	$-r^2(1-r^2)$	$2r^2-1$	1	$ds(\xi)$
11.	$\frac{1-r^2}{4}$	$\frac{1+r^2}{2}$	$\frac{1-r^2}{4}$	$nc(\xi) \pm sc(\xi)$ or $\frac{cn(\xi)}{1\pm sn(\xi)}$
12.	$\frac{-(1-r^2)^2}{4}$	$\frac{1+r^2}{2}$	$\frac{-1}{4}$	$rcn(\xi) \pm dn(\xi)$
13.	$\frac{1}{4}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$	$\frac{sn(\xi)}{1\pm cn(\xi)}$
14.	$\frac{1}{4}$	$\frac{1+r^2}{2}$	$\frac{(1-r^2)^2}{4}$	$\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$

Table 2

No.	*Symbols*	*Functions*	$r \to 0$	$r \to 1$
1.	$cn(\xi)$	$cn(\xi, r)$	$\cos(\xi)$	$\text{sech}(\xi)$
2.	$sn(\xi)$	$sn(\xi, r)$	$\sin(\xi)$	$\tanh(\xi)$
3.	$sc(\xi)$	$sc(\xi, r)$	$\tan(\xi)$	$\sinh(\xi)$
4.	$cs(\xi)$	$cs(\xi, r)$	$\cot(\xi)$	$\text{csch}(\xi)$
5.	$ns(\xi)$	$ns(\xi, r)$	$\csc(\xi)$	$\coth(\xi)$
6.	$dn(\xi)$	$dn(\xi, r)$	1	$\text{sech}(\xi)$
7.	$sd(\xi)$	$sd(\xi, r)$	$\sin(\xi)$	$\sinh(\xi)$
8.	$cd(\xi)$	$cd(\xi, r)$	$\cos(\xi)$	1
9.	$ds(\xi)$	$ds(\xi, r)$	$\csc(\xi)$	$\text{csch}(\xi)$
10.	$nc(\xi)$	$nc(\xi, r)$	$\sec(\xi)$	$\cosh(\xi)$

3. Solving (1) using aforementioned method

We make the following assumption to answer (1) using the ϕ^6-model expansion method:

$$u(x, y, z, t) = u(\xi), \qquad (12)$$
$$\xi = kx + ly + mz - wt$$

where $u(\xi)$ is a real function. After substituting (12) into (1), we get the following ordinary differential equation:

$$-wu(\xi) + \frac{A_1 k u^2(\xi)}{2} + \left(A_2 k^3 + A_3 k l^2 + A_3 k m^2\right) u''(\xi) = 0. \qquad (13)$$

Now balancing $u^2(\xi)$ and $u''(\xi)$ in (13), we get $M = 2$, hence (13) has the solution

$$u(\xi) = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3 + a_4\phi^4, \qquad (14)$$

where a_0, a_1, a_2, a_3 and a_4 are constants to be found later.

Inserting (14) along with (6) in (13) and equating the coefficients of like powers of $\phi^i(\xi)$ $(i = 0, 1, 2, ...8)$ to zero, we get the following algebraic equations:

$\phi^0 : 2a_2b_0A_2k^3 + \frac{1}{2}A_1ka_0^2 + 2a_2b_0A_3kl^2 + 2a_2b_0A_3km^2 - wa_0 = 0,$

$\phi^1 : k^3A_2a_1b_2 + 6k^3A_2a_3b_0 + kl^2A_3a_1b_2 + 6kl^2A_3a_3b_0 + km^2A_3a_1b_2 + 6km^2A_3a_3b_0 + kA_1a_0a_1 - wa_1 = 0,$

$\phi^2 : \frac{1}{2}A_1ka_1^2 + 4a_2b_2A_2k^3 + 12a_4b_0A_2k^3 + A_1ka_0a_2 - wa_2 + 4a_2b_2A_3kl^2 + 4a_2b_2A_3km^2 + 12a_4b_0A_3kl^2 + 12a_4b_0A_3km^2 = 0,$

$\phi^3 : 2k^3A_2a_1b_4 + 9k^3A_2a_3b_2 + 2kl^2A_3a_1b_4 + 9kl^2A_3a_3b_2 + 2km^2A_3a_1b_4 + 9km^2A_3a_3b_2 + kA_1a_0a_3 + kA_1a_1a_2 - wa_3 = 0,$

$\phi^4 : \frac{1}{2}A_1ka_2^2 + 6a_2b_4A_2k^3 + 16a_4b_2A_2k^3 + A_1ka_0a_4 + A_1ka_1a_3 - wa_4 + 6a_2b_4A_3kl^2 + 6a_2b_4A_3km^2 + 16a_4b_2A_3kl^2 + 16a_4b_2A_3km^2 = 0,$

$\phi^5 : 3k^3A_2a_1b_6 + 12k^3A_2a_3b_4 + 3kl^2A_3a_1b_6 + 12kl^2A_3a_3b_4 + 3km^2A_3a_1b_6 + 12km^2A_3a_3b_4 + kA_1a_1a_4 + kA_1a_2a_3 = 0,$

$\phi^6 : \frac{1}{2}A_1ka_3^2 + 8a_2b_6A_2k^3 + 20a_4b_4A_2k^3 + A_1ka_2a_4 + 20a_4b_4A_3km^2 + 8a_2b_6A_3kl^2 + 8a_2b_6A_3km^2 + 20a_4b_4A_3kl^2 = 0,$

$\phi^7 : 15k^3A_2a_3b_6 + 15kl^2A_3a_3b_6 + 15km^2A_3a_3b_6 + kA_1a_3a_4 = 0,$

$\phi^8 : \frac{1}{2}A_1ka_4^2 + 24a_4b_6A_2k^3 + 24a_4b_6A_3kl^2 + 24a_4b_6A_3km^2 = 0,$

(15)

Solving the algebraic system (15) by using Maple, we obtain the following results:

$$a_0 = a_0,\ a_1 = 0,\ a_2 = \frac{-12b_4\left((l^2+m^2)A_3 + k^2A_2\right)}{A_1},\ a_3 = 0,\ a_4 = 0,$$

$$b_0 = \frac{1}{48}\frac{(a_0kA_1 - 2w)\,a_0A_1}{\left((l^2+m^2)\,A_3 + k^2A_2\right)^2 kb_4},\ b_2 = \frac{1}{4}\frac{-a_0kA_1 + w}{\left((l^2+m^2)\,A_3 + k^2A_2\right)k},\ b_4 = b_4,\ b_6 = 0.$$

(16)

Using Eqs. (7), (14), and (16) as well as the Jacobi elliptic functions provided in the table above, we obtain the following exact solutions to (1):

3.1. If $c_0 = 1$, $c_2 = -\left(1 + r^2\right)$ and $c_4 = r^2$, thus $P(\xi) = sn(\xi)$ or $cd(\xi)$, and we obtained solutions listed below:

(17) $$u_1(x,y,z,t) = a_0 + a_2\frac{sn^2(\xi)}{fsn^2(\xi) + g},$$

or

(18) $$u_2(x,y,z,t) = a_0 + a_2\frac{cd^2(\xi)}{fcd^2(\xi) + g},$$

where our sources for f and g are

$$f=\frac{-b_4\left(1+b_2+r^2\right)}{\left(r^2+b_2+1\right)^2+3r^2-2r\left(r^2+b_2+1\right)^2-2},$$

$$g=\frac{3b_4}{\left(r^2+b_2+1\right)^2+3r^2-2r\left(r^2+b_2+1\right)^2-2},$$

under the condition of restriction

$$-b_4^2\left(1+b_2+r^2\right)\left(9r^2-\left(1+b_2+r^2\right)\left(2-b_2+2r^2\right)\right)=0,$$

where $\xi=kx+ly+mz-wt$.

In particular, if $r\to 1$, then we obtained solution listed below:

$$u_{1.1}(x,y,z,t)=a_0+a_2\frac{\tanh^2(\xi)}{f\tanh^2(\xi)+g}, \tag{19}$$

while, if $r\to 0$, then we obtained solutions listed below:

$$u_{1.2}(x,y,z,t)=a_0+a_2\frac{\sin^2(\xi)}{f\sin^2(\xi)+g}, \tag{20}$$

or

$$u_{2.2}(x,y,z,t)=a_0+a_2\frac{\cos^2(\xi)}{f\cos^2(\xi)+g}. \tag{21}$$

3.2. If $c_0=1-r^2$, $c_2=2r^2-1$ and $c_4=-r^2$, thus $P(\xi)=cn(\xi)$, and we obtained solution listed below:

$$u_3(x,y,z,t)=a_0+a_2\frac{cn^2(\xi)}{fcn^2(\xi)+g}, \tag{22}$$

where our sources for f and g are

$$f=\frac{b_4\left(-1-b_2+2r^2\right)}{\left(-1-b_2+2r^2\right)^2-3r^2\left(-r^2+1\right)-2\left(2r^2-1\right)\left(-1-b_2+2r^2\right)},$$

$$g=\frac{3b_4\left(-r^2+1\right)}{\left(-1-b_2+2r^2\right)^2-3r^2\left(-r^2+1\right)-2\left(2r^2-1\right)\left(-1-b_2+2r^2\right)},$$

under the condition of restriction

$$b_4^2\left(-1-b_2+2r^2\right)\left(-9r^2\left(-r^2+1\right)-\left(-1-b_2+2r^2\right)\left(-2+b_2+4r^2\right)\right)=0,$$

where $\xi=kx+ly+mz-wt$.

3.3. If $c_0 = r^2 - 1$, $c_2 = 2 - r^2$ and $c_4 = -1$, thus $P(\xi) = dn(\xi)$, and we obtained solution listed below:

$$u_4(x, y, z, t) = a_0 + a_2 \frac{dn^2(\xi)}{f dn^2(\xi) + g}, \tag{23}$$

where our sources for f and g are

$$f = \frac{-b_4\left(-2 + b_2 + r^2\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

$$g = \frac{3b_4\left(r^2 - 1\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

under the condition of restriction

$$-b_4^2\left(-2 + b_2 + r^2\right)\left(-9r^2 + 9 + \left(-2 + b_2 + r^2\right)\left(4 + b_2 - 2r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

3.4. If $c_0 = r^2$, $c_2 = -\left(1 + r^2\right)$ and $c_4 = 1$, thus $P(\xi) = ns(\xi)$ or $dc(\xi)$, and we obtained solutions listed below:

$$u_5(x, y, z, t) = a_0 + a_2 \frac{ns^2(\xi)}{f ns^2(\xi) + g}, \tag{24}$$

or

$$u_6(x, y, z, t) = a_0 + a_2 \frac{dc^2(\xi)}{f dc^2(\xi) + g}, \tag{25}$$

where our sources for f and g are

$$f = \frac{-b_4\left(1 + b_2 + r^2\right)}{\left(r^2 + b_2 + 1\right)^2 + 3r^2 - 2r\left(r^2 + b_2 + 1\right)^2 - 2},$$

$$g = \frac{3r^2 b_4}{\left(r^2 + b_2 + 1\right)^2 + 3r^2 - 2r\left(r^2 + b_2 + 1\right)^2 - 2},$$

under the condition of restriction

$$-b_4^2\left(1 + b_2 + r^2\right)\left(9r^2 - \left(1 + b_2 + r^2\right)\left(2 - b_2 + 2r^2\right)\right) = 0$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 1$, then we obtained solution listed below:

$$u_{5.1}(x, y, z, t) = a_0 + a_2 \frac{\coth^2(\xi)}{f \coth^2(\xi) + g}. \tag{26}$$

$$u_{6.1}(x, y, z, t) = a_0 + a_2 \frac{\operatorname{csch}^2(\xi)}{f \operatorname{csch}^2(\xi) + g}. \tag{27}$$

3.5. If $c_0 = -r^2$, $c_2 = 2r^2 - 1$ and $c_4 = 1 - r^2$, thus $P(\xi) = nc(\xi)$, and we obtained solution listed below:

$$u_7(x, y, z, t) = a_0 + a_2 \frac{nc^2(\xi)}{fnc^2(\xi) + g}, \tag{28}$$

where our sources for f and g are

$$f = \frac{b_4\left(-1 - b_2 + 2r^2\right)}{\left(-1 - b_2 + 2r^2\right)^2 - 3r^2\left(-r^2 + 1\right) - 2\left(2r^2 - 1\right)\left(-1 - b_2 + 2r^2\right)},$$

$$g = \frac{-3r^2 b_4}{\left(-1 - b_2 + 2r^2\right)^2 - 3r^2\left(-r^2 + 1\right) - 2\left(2r^2 - 1\right)\left(-1 - b_2 + 2r^2\right)},$$

under the condition of restriction

$$b_4^2\left(-1 - b_2 + 2r^2\right)\left(-9r^2\left(-r^2 + 1\right) - \left(-1 - b_2 + 2r^2\right)\left(-2 + b_2 + 4r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 1$, then we obtained solution listed below:

$$u_{7.1}(x, y, z, t) = a_0 + a_2 \frac{\cosh^2(\xi)}{f \cosh^2(\xi) + g}. \tag{29}$$

3.6. If $c_0 = -1$, $c_2 = 2 - r^2$ and $c_4 = -\left(1 - r^2\right)$, thus $P(\xi) = nd(\xi)$, and we obtained solution listed below:

$$u_8(x, y, z, t) = a_0 + a_2 \frac{nd^2(\xi)}{fnd^2(\xi) + g}, \tag{30}$$

where our sources for f and g are

$$f = \frac{-b_4\left(-2 + b_2 + r^2\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

$$g = \frac{-3b_4}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

under the condition of restriction

$$-b_4^2\left(-2 + b_2 + r^2\right)\left(-9r^2 + 9 + \left(-2 + b_2 + r^2\right)\left(4 + b_2 - 2r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

3.7. If $c_0 = 1$, $c_2 = 2 - r^2$ and $c_4 = 1 - r^2$, thus $P(\xi) = sc(\xi)$, and we obtained solution listed below:

$$u_9(x, y, z, t) = a_0 + a_2 \frac{sc^2(\xi)}{fsc^2(\xi) + g}, \tag{31}$$

where our sources for f and g are

$$f = \frac{-b_4\left(-2 + b_2 + r^2\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

$$g = \frac{3b_4}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

under the condition of restriction

$$-b_4^2\left(-2 + b_2 + r^2\right)\left(-9r^2 + 9 + \left(-2 + b_2 + r^2\right)\left(4 + b_2 - 2r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 1$, then we obtained solution listed below:

$$u_{9.1}(x, y, z, t) = a_0 + a_2 \frac{\sinh^2(\xi)}{f \sinh^2(\xi) + g}, \tag{32}$$

while, if $r \to 0$, then we obtained solution listed below:

$$u_{9.2}(x, y, z, t) = a_0 + a_2 \frac{\tan^2(\xi)}{f \tan^2(\xi) + g}. \tag{33}$$

3.8. If $c_0 = 1$, $c_2 = 2r^2 - 1$ and $c_4 = -r^2\left(1 - r^2\right)$, thus $P(\xi) = sd(\xi)$, we obtained solution listed below:

$$u_{10}(x, y, z, t) = a_0 + a_2 \frac{sd^2(\xi)}{f sd^2(\xi) + g}, \tag{34}$$

where our sources for f and g are

$$f = \frac{b_4\left(-1 - b_2 + 2r^2\right)}{\left(-1 - b_2 + 2r^2\right)^2 - 3r^2\left(-r^2 + 1\right) - 2\left(2r^2 - 1\right)\left(-1 - b_2 + 2r^2\right)},$$

$$g = \frac{3b_4}{\left(-1 - b_2 + 2r^2\right)^2 - 3r^2\left(-r^2 + 1\right) - 2\left(2r^2 - 1\right)\left(-1 - b_2 + 2r^2\right)},$$

under the condition of restriction

$$b_4^2\left(-1 - b_2 + 2r^2\right)\left(-9r^2\left(-r^2 + 1\right) - \left(-1 - b_2 + 2r^2\right)\left(-2 + b_2 + 4r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

3.9. If $c_0 = 1 - r^2$, $c_2 = 2 - r^2$ and $c_4 = 1$, thus $P(\xi) = cs(\xi)$, and we obtained solution listed below:

$$u_{11}(x, y, z, t) = a_0 + a_2 \frac{cs^2(\xi)}{f cs^2(\xi) + g}, \tag{35}$$

where our sources for f and g are

$$f = \frac{-b_4\left(-2 + b_2 + r^2\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

$$g = \frac{3b_4\left(-r^2 + 1\right)}{\left(r^2 + b_2 - 2\right)^2 - 3r^2 + 7 - 2r\left(r^2 + b_2 - 2\right)^2},$$

under the condition of restriction

$$-b_4^2\left(-2 + b_2 + r^2\right)\left(-9r^2 + 9 + \left(-2 + b_2 + r^2\right)\left(4 + b_2 - 2r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 0$, then we obtained solution listed below:

$$u_{11.2}(x, y, z, t) = a_0 + a_2 \frac{\cot^2(\xi)}{f \cot^2(\xi) + g}. \tag{36}$$

3.10. If $c_0 = -r^2\left(1-r^2\right)$, $c_2 = 2r^2-1$ and $c_4 = 1$, thus $P(\xi) = ds(\xi)$, and we obtained solution listed below:

$$u_{12}(x,y,z,t) = a_0 + a_2\frac{ds^2(\xi)}{fds^2(\xi)+g}, \tag{37}$$

where our sources for f and g are

$$f = \frac{b_4\left(-1-b_2+2r^2\right)}{\left(-1-b_2+2r^2\right)^2 - 3r^2\left(-r^2+1\right) - 2\left(2r^2-1\right)\left(-1-b_2+2r^2\right)},$$

$$g = \frac{3r^2 b_4\left(r^2-1\right)}{\left(-1-b_2+2r^2\right)^2 - 3r^2\left(-r^2+1\right) - 2\left(2r^2-1\right)\left(-1-c_2+2r^2\right)},$$

under the condition of restriction

$$b_4^2\left(-1-b_2+2r^2\right)\left(-9r^2\left(-r^2+1\right) - \left(-1-b_2+2r^2\right)\left(-2+b_2+4r^2\right)\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

3.11. If $c_0 = \frac{1-r^2}{4}$, $c_2 = \frac{1+r^2}{2}$ and $c_4 = \frac{1-r^2}{4}$, thus $P(\xi) = nc(\xi) \pm sc(\xi)$ or $\frac{cn(\xi)}{1\pm sn(\xi)}$, and we obtained the solutions listed below:

$$u_{13}(x,y,z,t) = a_0 + a_2\frac{(nc(\xi)\pm sc(\xi))^2}{f(nc(\xi)\pm sc(\xi))^2+g}, \tag{38}$$

or

$$u_{14}(x,y,z,t) = a_0 + a_2\frac{cn^2(\xi)}{fcn^2(\xi)+g(1\pm sn(\xi))^2}, \tag{39}$$

where our sources for f and g are

$$f = \frac{-8b_4\left(1-2b_2+r^2\right)}{r^4+14r^2-16b_2^2+1},$$

$$g = \frac{12b_4\left(r^2-1\right)}{r^4+14r^2-16b_2^2+1},$$

under the condition of restriction

$$\frac{1}{32}b_4^2\left(1-2b_2+r^2\right)\left(r^4+8r^2b_2-34r^2+16b_2^2+8b_2+1\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 0$, then we obtained the solutions listed below:

$$u_{13.2}(x,y,z,t) = a_0 + a_2\frac{(\sec(\xi)\pm\tan(\xi))^2}{f(\sec(\xi)\pm\tan(\xi))^2+g}, \tag{40}$$

or

$$u_{14.2}(x,y,z,t) = a_0 + a_2\frac{\cos^2(\xi)}{f(\cos^2(\xi))+g(1\pm\sin(\xi))^2}. \tag{41}$$

3.12. If $c_0 = \frac{-\left(1-r^2\right)^2}{4}$, $c_2 = \frac{1+r^2}{2}$ and $c_4 = \frac{-1}{4}$, thus $P(\xi) = rcn(\xi) \pm dn(\xi)$, and we obtained the solution listed below:

$$u_{15}(x,y,z,t) = a_0 + a_2 \frac{\left(rcn(\xi) \pm dn(\xi)\right)^2}{f\left(rcn(\xi) \pm dn(\xi)\right)^2 + g}, \tag{42}$$

where our sources for f and g are

$$f = \frac{-8b_4\left(1 - 2b_2 + r^2\right)}{r^4 + 14r^2 - 16b_2^2 + 1},$$

$$g = \frac{12b_4\left(r^2 - 1\right)^2}{r^4 + 14r^2 - 16b_2^2 + 1},$$

under the condition of restriction

$$\frac{1}{32}b_4^2\left(1 - 2b_2 + r^2\right)\left(r^4 + 8r^2b_2 - 34r^2 + 16b_2^2 + 8b_2 + 1\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

3.13. If $c_0 = \frac{1}{4}$, $c_2 = \frac{1-2r^2}{2}$ and $c_4 = \frac{1}{4}$, thus $P(\xi) = \frac{sn(\xi)}{1 \pm cn(\xi)}$, and we obtained solution listed below:

$$u_{16}(x,y,z,t) = a_0 + a_2 \frac{sn^2(\xi)}{fsn^2(\xi) + g\left(1 \pm cn(\xi)\right)^2}, \tag{43}$$

where our sources for f and g are

$$f = \frac{8b_4\left(-1 + 2b_2 + 2r^2\right)}{16r^4 - 16r^2 - 16b_2^2 + 1},$$

$$g = \frac{-12b_4}{16r^4 - 16r^2 - 16b_2^2 + 1},$$

under the condition of restriction

$$\frac{1}{32}b_4^2\left(-1 + 2b_2 + 2r^2\right)\left(32r^4 + 16r^2b_2 - 32r^2 - 16b_2^2 - 8b_2 - 1\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

In particular, if $r \to 1$, then we obtained the solution listed below:

$$u_{16.1}(x,y,z,t) = a_0 + a_2 \frac{\tanh^2(\xi)}{f\tanh^2(\xi) + g\left(1 \pm \operatorname{sech}(\xi)\right)^2}, \tag{44}$$

while, if $r \to 0$, then we obtained the solution listed below:

$$u_{16.2}(x,y,z,t) = a_0 + a_2 \frac{\sin^2(\xi)}{f\sin^2(\xi) + g\left(1 \pm \cos(\xi)\right)^2}. \tag{45}$$

3.14. If $c_0 = \frac{1}{4}$, $c_2 = \frac{1+r^2}{2}$ and $c_4 = \frac{\left(1-r^2\right)^2}{4}$, thus $P(\xi) = \frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$, and we obtained solution listed below:

$$u_{17}(x,y,z,t) = a_0 + a_2 \frac{sn^2(\xi)}{f sn^2(\xi) + g\left(cn(\xi) \pm dn(\xi)\right)^2}, \tag{46}$$

where our sources for f and g are

$$f = \frac{-8b_4\left(1-2b_2+r^2\right)}{r^4+14r^2-16b_2^2+1},$$

$$g = \frac{-12b_4}{r^4+14r^2-16b_2^2+1},$$

under the condition of restriction

$$\frac{1}{32}c_4^2\left(1-2c_2+r^2\right)\left(r^4+8r^2c_2-34r^2+16c_2^2+8c_2+1\right) = 0,$$

where $\xi = kx + ly + mz - wt$.

4. Graphical representation of some obtained solutions

To effectively illustrate the behaviour of the solutions, a few representative plots of the resolved problems are presented below.

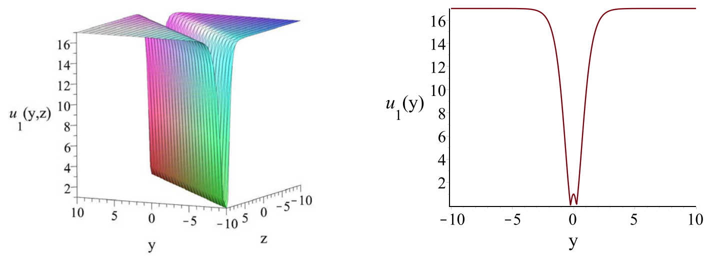

Figure 1. Three dimensional and two dimensional graphs of u_1 for various choice of parameters.

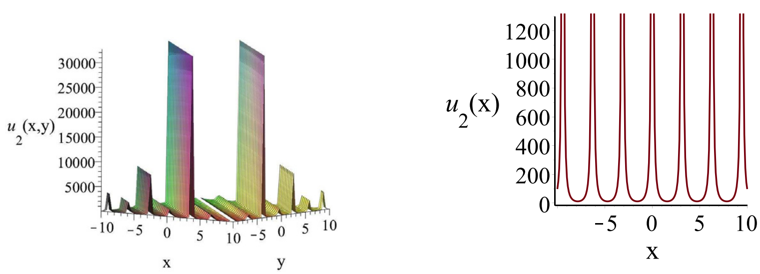

Figure 2. Three dimensional and two dimensional graphs of u_2 for various choice of parameters.

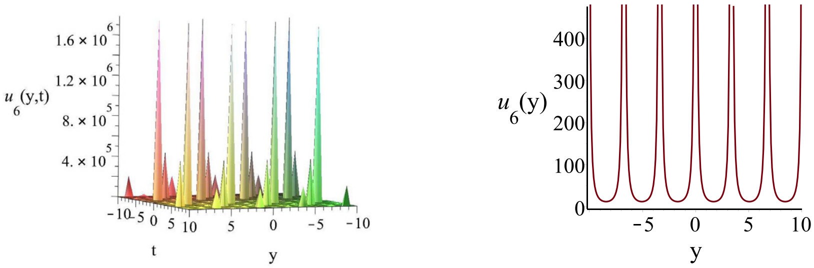

FIGURE 3. Three dimensional and two dimensional graphs of u_6 for various choice of parameters.

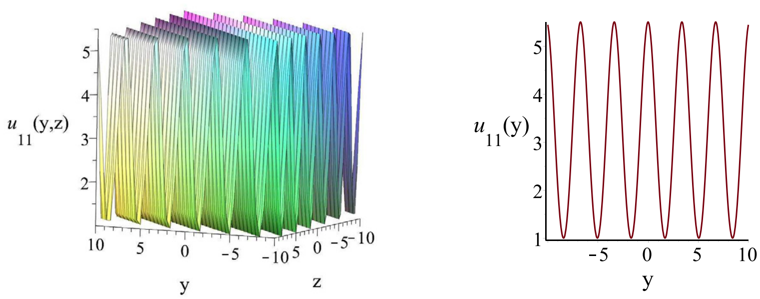

FIGURE 4. Three dimensional and two dimensional graphs of u_{11} for various choice of parameters.

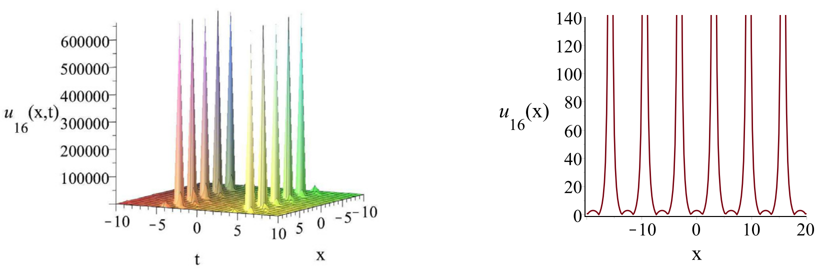

FIGURE 5. Three dimensional and two dimensional graphs of u_{16} for various choice of parameters.

5. Result and discussion

This section contains the ZK equation's exact solutions and their graphical representations. By directly comparing our findings with those presented in [3, 15, 18], we have successfully derived several novel exact solutions. The obtained solutions consist of trigonometric, hyperbolic, and Jacobi elliptic function-based solutions. We further illustrate the validity and richness of the solutions by visualizing them through 3D and 2D plots, which are created by choosing suitable parameter values that meet the constraint requirement. We employ Maple, a powerful mathematical software, to generate graphical representations of these solutions. We present a brief analysis of the solution dynamics below.

Figure 1 displays the W-shaped soliton for the solution $u_1(x, y, z, t)$ with appropriate values of parameters, $x = 0$, $t = 0$, $r = 1$, $A_1 = 6$, $A_2 = 3$, $A_3 = 3$, $k = l = m = 1$, $a_0 = 1$, $b_4 = 1$, $w = -66$. These soliton solutions typically represent a localized and stable wave structure. The W-shaped profile reflects how energy distributes over two regions while maintaining the soliton's stability and localized propagation characteristic. Figure 2 displays the solution's dynamic behavior with a periodic soliton structure for the solution $u_2(x, y, z, t)$ for the appropriate values of parameters, $z = 0$, $t = 0$, $r = 0$, $A_1 = 6$, $A_2 = 3$, $A_3 = 3$, $k = l = m = 1$, $a_0 = 5$, $b_4 = 1$, $w = 102$. Periodic soliton solutions are unique soliton solutions that exhibit spatially or temporally periodic behavior rather than decaying to zero at infinity like traditional solitons. These solutions represent waves that propagate with stable and periodic structures. Figure 3 displays the solution's dynamic behavior with a singular soliton structure for the solution $u_6(x, y, z, t)$ for the appropriate values of parameters, $x = 0$, $z = 0$, $r = \frac{1}{2}$, $A_1 = 6$, $A_2 = 3$, $A_3 = 3$, $k = l = m = 1$, $a_0 = 1$, $b_4 = 3$, $w = -12$. Singular soliton solutions are a special wave solution characterized by singularities such as infinite amplitude, acute cusps, or discontinuities at certain places. Unlike smooth and localized regular solitons, these solutions defy traditional expectations while maintaining remarkable stability, propagating consistently under the equation's governing dynamics. Their persistence highlights the delicate balance of non-linearity and dispersion in complex systems. Figure 4 displays the solution's dynamic behavior with a periodic soliton structure for the solution $u_{11}(x, y, z, t)$ for the appropriate values of parameters, $x = 0$, $t = 0$, $r = \frac{1}{2}$, $A_1 = 6$, $A_2 = 3$, $A_3 = 3$, $k = l = m = 1$, $a_0 = 1$, $b_4 = -3$, $w = -12$. Figure 5 displays solution's dynamic behavior with a singular soliton structure for the solution $u_{16}(x, y, z, t)$ for the appropriate values of parameters, $y = 0$, $z = 0$, $r = 0$, $A_1 = 1$, $A_2 = 1$, $A_3 = 1$, $k = l = m = 1$, $a_0 = 3$, $b_4 = 1$, $w = 9$.

6. Conclusion

We address the nonlinear $(3 + 1)$-dimensional ZK equation using a recently discovered ϕ^6-model expansion technique and obtain novel travelling wave solutions to the said equation in the form of Jacobi's elliptic functions, which can be valuable for researchers in studying and understanding the physical interpretation of the ZK equation. The obtained solutions approach trigonometric solutions when $r \to 0$ and hyperbolic solutions when $r \to 1$. These results indicate that the employed method has produced numerous novel and significant solutions to the considered equation. Furthermore, we visualize specific solutions using $3-$D and $2-$D graphs.

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